Translated from: Harmonik & Glasperlenspiel. Beiträge `93. München 1994

What does 'just intonation' mean?

Peter Neubäcker

The lecture I would like to deliver today is the first part of a symposium on 'just temperaments' that we will continue tomorrow - but this lecture can also be seen as completely independent of it. The question today is: What, if anything, is 'just intonation'? Why is there such a thing, why do people talk about it, what do they aim to accomplish by it? Then tomorrow the subject will switch to somewhat more specific themes, such as the questions: 'What aspects do the just temperaments have?' and 'How can we realize these just temperaments in practice?' First, then, the question: What is 'just intonation'?

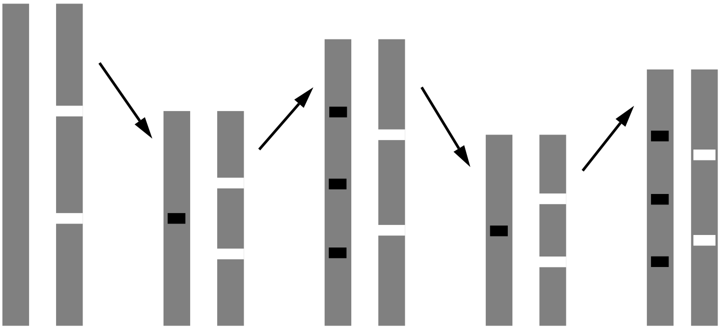
Often the question of tuning is not raised at all. For those who play the piano, for example, it's simply like this: There are white keys and there are black keys, and they just play, correctly or incorrectly, as they have learnt. When someone is no longer satisfied with the tuning, they say: "The piano needs retuning ", and the piano tuner arrives and tunes it again. The basic assumption in such cases is that the piano is either 'correctly' tuned or out of tune - the idea that there might be several, or even many, 'correct' tunings is something of which most musicians are ignorant. (Or, in fact, in the case of the piano, no 'correct' tuning at all but simply a compromise suitable for a particular purpose - but we'll see that later).

It is different, for example, if someone is learning Indian music: there he learns that a tone which on our piano is represented by a certain key must be taken somewhat higher in a certain context, such as in a particular raga, and in another somewhat lower. Here, then, 'correct' and 'incorrect' depend upon the context in which the tone is found. And, in fact, that is also the case in our music - it is simply that we are not consciously trained to be as attentive towards it. We know that there are twelve tones in an octave, and somehow we come to terms with that fact - like it little or like it a lot. The tuning is simply something given, and scant thought is devoted to the question of whether it has to be the way it is or whether it could also be otherwise.

The question of 'just intonation' therefore presents itself as a whole complex of questions - such as these: What does 'just' mean? Measured according to which system? Is there a difference between 'just' and 'correct'? Is it objective at all or is it subjective? Why should tuning be just anyway? Can it also be totally different? Is a tuning system simply a cultural convention or is it something prescribed by Nature? I would like perhaps to anticipate at this stage by putting it this way: In the area where the term 'correct' is used, we are dealing with a system that stands in a context of culture and convention - this has a subjective character. In the area where the term 'just' is used, on the other hand, we are dealing with a system that stands in a context of auditory physiology and the physical and mathematical fundamentals of music - something that is given by Nature, in other words, and has an objective character. It is about this second area that I will primarily be speaking this evening.

First, though, let us ask ourselves 'what is a tone system anyway?' This question is already ancient. And the oldest tone system of which I am aware, and of the genesis of which a detailed account has survived, is an old Chinese system. Of course, much of the Chinese documentation has been lost - there have been many book burnings over the course of time - but some things have been preserved, and what is essentially of relevance in this context is to be found in the book The *Annals of Lü Buwei*. This tells how a legendary emperor of ancient times once assigned to his principal court musician the task of standardizing music, of discovering - extracting from Nature, so to speak - a system of music. And this man went into the mountains to the source of the Yellow River. There, he cut a length of bamboo between two knots, blew into it, and said: "*This is in right*". The tone was in tune with his voice when he spoke without affect and in tune with the murmuring of the spring there that is the source of the Yellow River. Then, while he was lost in contemplation, the Phoenix happened by with his mate. The pair then sang to him twelve tones; that is to say, the male phoenix sang six tones and the female phoenix sang six more. There were therefore male and female tones, yin and yang tones. Then he cut further bamboo pipes to match these tones and returned home with them to the imperial palace. There, bells were cast, and the whole thing became the musical system.

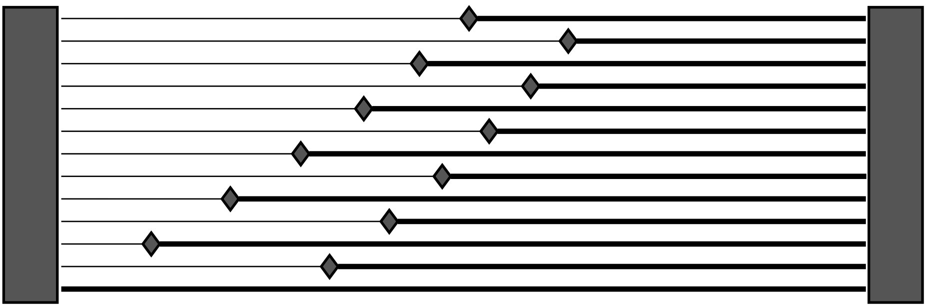
Later comes a description of what this system looked like. It was described with great precision - that is to say, in mathematical terms - how the system was implemented. And the algorithm went like this: I take a bamboo tube of a given length - I'll draw it here - and from this bamboo tube, I take away a third; that gives me the length of the next bamboo tube. I discover the length of the bamboo tube that follows by dividing the second one, again, into thirds; but this time, instead of taking one of these thirds away, I add another the same length. That is obviously not the same thing as adding the third I took from the first tube: the third tube is somewhat shorter than the first. Now I continue in the same fashion as before: I take a third away from the new tube, and so it continues in alternation - each time, after taking away a third, I add a third of the new length. One could carry on like this for ever in theory, but here we'll just draw the progression as far as the fifth tube. You have each time a longer tube followed by a shorter one but with the overall tendency a steady reduction.



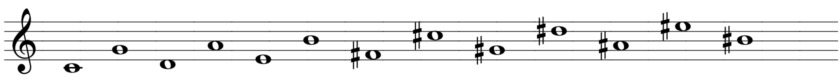
The question is now: Why do I take away a third in the first case and add a third in the second? The reason is simply a practical one: the tubes would otherwise very quickly become too short and I would end up with one so tiny that it would be impossible to get a sound out of it; we don't want to arrive too soon at the stage where the tubes are too short to play. But what justifies us - if what we really want to do is subtract a third each time - in adding a third instead on every other occasion? Let's look and see what the difference is between subtracting a third and adding one: we have, in the first case, 3/3 - 1/3 = 2/3, and in the second, 3/3 + 1/3 = 4/3. The result, then, in the first case is 2/3 and in the second 4/3 - which means that the result of the second operation is exactly double that of the first.

Here we arrive at a fundamental phenomenon: I have here the string of a monochord, and beneath the string, exactly at its midpoint, I place a bridge - effectively halving the length of the string. Now, if we compare the tone produced by the string vibrating along its entire length with that produced by this half, the resulting interval is what is called in musical terms an 'octave'. This operation of the number *Two* - mathematically the simplest operation I can perform (halving or doubling) - leads us, then, to an extraordinarily weird musical experience: I obtain two tones and have the impression that they are the same, even though they are quite different and are in fact a long way apart in pitch. We give the same note name - 'C', for example - to all tones at octave intervals from one another. And because it is so weird and at the same time so simple, we are scarcely able to wonder at it.

And it is precisely this phenomenon that justifies us with the bamboo tubes in subtracting 1/3 in one case and adding 1/3 in the other. Whether I double a given length or halve it, I arrive at the same tone - that is to say, at a tone I call by the same name. The operation described in this Chinese system is based upon nothing other than the number *Three*, whereby the *Two* is assumed. So the number *Two* is something we take for granted, the primordial phenomenon, and the number *Three* - the division into three parts - is the secondary (if you like, the 'second most primordial') phenomenon, and the interval that it gives us is what we call in musical terms a 'fifth'. Here I have set up the monochord to reproduce the intervals we get if we implement the Chinese algorithm described above. *[In the illustration, the rightmost section of each string (where the line is thicker) is the part that is sounding]*



Following the algorithm here, I have generated twelve new tones to obtain the following sequence - let's hear how it sounds:



Note *on the display of tones in the written form of this lecture: It is in fact totally false to represent tones in this context as notes, since to do so presupposes that the notes and the tone system already exist. In reality what we are doing here is demonstrating phenomena that predated, and exist fully independently of, any tone system. This lecture is about experiencing the pure phenomenon outside of any cultural-musical context. Even the fundamental, C, has been arbitrarily chosen. The notes, therefore, only serve here to allow readers to play the tones through - on a piano, for example - in order to form an impression of what is meant, before (ideally) forgetting at once that they took the tones from the notes. Furthermore, the impression given by the tones written down and heard on the piano does not entirely correspond to the string division, since the piano is tuned somewhat differently. It would be best, then, to perform the procedure on a monochord.*

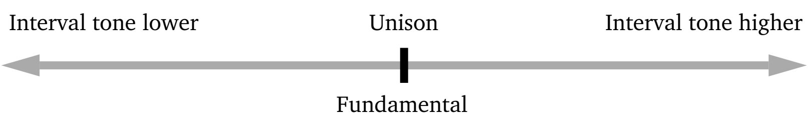
The question is now: How far does it go? Is there a limit or can you just carry on like this forever? Theoretically, of course, you can. I have stopped here after the first 12 steps and done so for a particular reason: Let us listen again to the sound of the different intervals yielded by this operation; this time, though, at the same time as each of the new tones, I will sound a second: that produced by the entire string, the fundamental. We will see that with each new harmonic interval, i.e. with each pair of tones sounding simultaneously, we get a different listening experience, we are aware of different interval qualities - until, that is, we arrive at the thirteenth string, where we suddenly have the impression that we are hearing the first tone in the series again, only an octave higher. It is, however, slightly out of tune. So, whilst with the first twelve strings each combination yields a different interval quality, when we arrive at the thirteenth, we have the impression that there is nothing new, but rather that the qualities are repeating: that we have completed a circle. Except, though, as we have heard, the circle does not close completely: the thirteenth tone is slightly out of tune, if we were to compare it with a tone exactly an octave above our starting tone. The cause is not a failure on my part to tune the monochord correctly; the discrepancy is an inevitable product of the system. We could say that the circle closes on a musical-qualitative level but that it does not close on a mathematical-quantitative one. To make the whole thing clear from a computational standpoint: We obtained the lengths of our bamboo tubes by dividing repeatedly by three and then 'octaving', i.e. multiplying by 2 or 4 to obtain a usable length. We can represent the string lengths, then, by using fractions in which the denominator is each time a power of three and the numerator the compensating power of two:

Power-Ratio	$\frac{2^0}{3^0}$	$\frac{2^1}{3^1}$	$\frac{2^3}{3^2}$	$\frac{2^4}{3^3}$	$\frac{2^6}{3^4}$	$\frac{2^7}{3^5}$
Fraction	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{16}{27}$	$\frac{64}{81}$	$\frac{128}{243}$
Monochord 120 cm	120	80	106.66	71.11	94.81	63.20

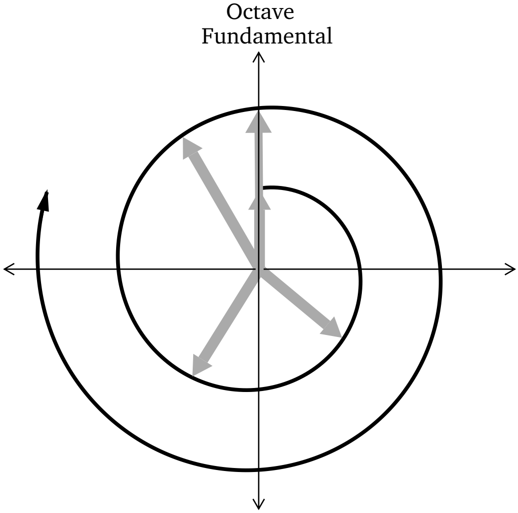
$\frac{2^9}{3^6}$	$\frac{2^{10}}{3^7}$	$\frac{2^{12}}{3^8}$	$\frac{2^{13}}{3^9}$	$\frac{2^{15}}{3^{10}}$	$\frac{2^{16}}{3^{11}}$	$\frac{2^{18}}{3^{12}}$
$\frac{512}{729}$	$\frac{1024}{2187}$	$\frac{4096}{6561}$	$\frac{8192}{19683}$	$\frac{32768}{59049}$	$\frac{65536}{177147}$	$\frac{262144}{531441}$
84.27	56.19	74.92	49.94	66.59	44.39	59.19

In the bottom row, we have here the string lengths that I have set on the monochord. We see that the last step yields a string length of 59.19 cm - this last string is therefore some 8 mm shorter than the octave at 60 cm. We hear this difference as the octave being out of tune, but not as a new quality of the kind we have experienced with the preceding steps.

We wish now to illustrate the whole thing by means of a further graphic. To do this, we must think about how we can sensibly depict these facts. One possibility would be a linear depiction of the distance from one tone to another, in this case from the interval tone to the fundamental.



Here, then, we have simply the physical distance from one tone to the next. Our experience, however, is different: As one tone distances itself continuously from another, at first, admittedly, we perceive it as distancing, but as the second tone approaches double the frequency of the first, we perceive this as a return to the first tone, which is just what we describe as the experience of the octave. Through the octave or the number *Two*, then, the tonal space is cyclically structured. We experience, as the second tone continuously rises, ever changing interval qualities that repeat, however, each time the number of vibrations doubles. To display that graphically, we need to introduce a second dimension, the direction:



The continuously rising tone is represented here by a spiral - the length of the arrows shows us (as in the previous diagram) the absolute distance from the interval tone to the fundamental tone - additionally, however, the arrow has a direction that indicates the quality of our experience of this interval. So the directions of the arrows in this illustration give approximately the experience of the fifth-like, of the third-like, of the seventh-like, and so on, whereby we have not yet defined here how exactly the direction is related to the interval experience. Now let us ask which directions precisely in this circle correspond to which interval qualities. Let us look first for the fifth, the ratio 3:2. We might perhaps suppose at first that the fifth must lie directly opposite the fundamental at 180°, since 3:2 = 1.5, a value that lies halfway between 1 and 2. But we also see at once that that cannot be correct: if it were, the complement of the fifth (i.e. the interval required to complete the octave) would be another fifth, since the corresponding angle would be another half circle - we know, however, that the complement of a fifth is not another fifth but a fourth 4:3. The correct angle for the fifth must, then, be somewhat larger than 180° if identical angles are to yield identical interval experiences. But how large must this angle be? Let us look again at how the frequencies, the cycle ratios of two tones, are related to the interval experience. In the case of the octave, it looks like this:

1	=	2^0	Fundamental	
2	=	2^1	1st octave	
4	=	2^2	2nd octave	
8	=	2^3	3rd octave	etc.

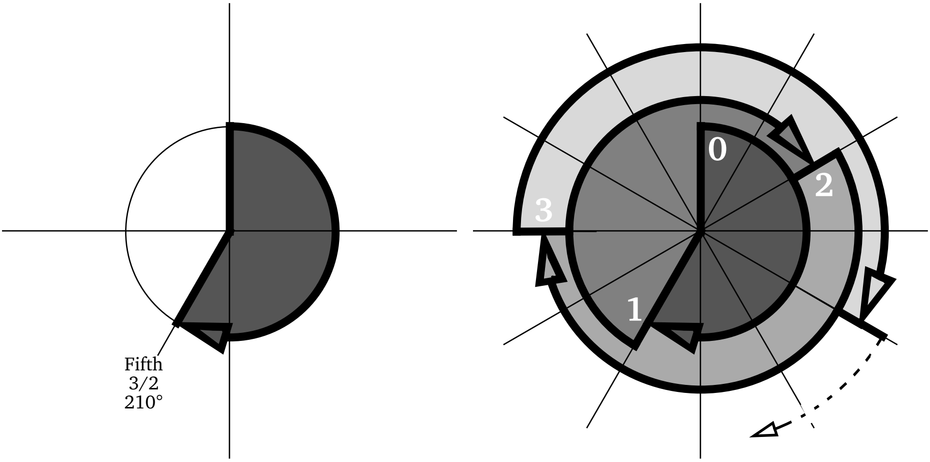
We see, then, on the left-hand side numbers representing the repeated doubling of the number of vibrations whilst on the right-hand side we see our qualitative experience of the successive octaves expressed in terms of powers of two - which, in the case of our circle, is equivalent to the number of completed turns - i.e. multipliers of 360°. This means that if we wish to determine the angle for a fifth, we must ask what power of three is needed to obtain the vibration ratio 3:2 = 1.5. Mathematically expressed, it looks like this:

$$2^{\text{fifth}} = 1.5$$

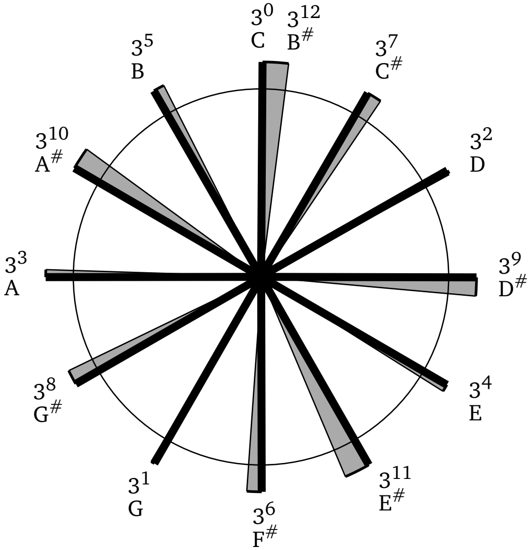
To find the value for the fifth, we therefore need to express both sides of the equation as logarithms, like this:

$$\log(2) \cdot \text{fifth} = \log(1.5) \quad \text{or}$$
$$\text{fifth} = \log(1.5) / \log(2) \quad \text{which gives us}$$
$$\text{fifth} = 0.176... / 0.301...$$
$$\text{fifth} = 0.5849625...$$

We have established, then, that the direction of the fifth in our illustration is equivalent to 0.5849625... times a full circle - in other words, 0.5849625 • 360° = 210.5865°. The angle of the fifth is almost exactly equal to 210° - which is a multiple of 30°, a twelfth of a circle. If we wish to enter the fifth continuously in our circular system of interval-quality-directions, we must therefore keep adding this value to the preceding one, to find the directions of the succeeding interval qualities:



We discover, then, that the operation of the fifth or the number three divides the circle of interval qualities into *twelve* directions - one could say, then, that the number *Three* structures the tonal space and creates twelve tone locations that are experienced as different interval qualities. We must, of course, distinguish here between the qualitative directions of the tone locations, which I described previously as '*fifth-like*', '*third-like*' etc. and, on the other side, the exact values of these tone locations, which arise from the continuous fifth operation 3:2. With the division into twelfths, I have ignored the fact that the precise value of the fifth is 210.5965˚ - which in the case of the individual fifths does not play that great a role, but in the system adds up to an ever increasing discrepancy. Here, I have illustrated how it would really look:



The darker lines indicate the exact division into twelve of the circle, and the grey wedges the deviation of the 'pure' tone created by the repeated addition of the pure fifth 3:2. Incidentally, I have entered note names here, with C at the top being the fundamental. We must bear in mind here, though, that the note names denote not absolute pitches, not, for example, as fundamental, the tone C, as determined by the piano, but simply an arbitrary relative reference tone, from which we can more easily name the other tones.

We see, then, that in the case of the first fifth, G (3^1), the deviation is relatively small, but that it becomes ever greater in the case of the fifths that follow. Using the diagram, we can always find the following fifth by applying the rule '*opposite and one along*' - until we arrive back at the top with a deviation of almost a quarter of the twelfth of a circle. Here, I have no longer even termed this tone 'C' but rather B#, which would be the correct term for it in the circle of fifths. This angle corresponds, then, exactly to the deviation we encountered earlier in the table as the difference between the octave 1/2 and the twelve fifths $3^{12}/2^{18}$.

This is perhaps the place to introduce a new unit of measurement: the *cent*. Admittedly, as we have seen, the proportion-numbers are the way of describing intervals that are best in keeping with their nature, but we do nonetheless need an easily understood unit of measurement for comparing intervals with one another, since one cannot without more look at the mutual size ratio of the proportions in connection with the interval perception. We have also seen that from the aural standpoint the correct comparison of interval sizes must proceed by raising the number *Two* to various powers: it is, in other words, a logarithmic unit of measurement. Such a unit is the *cent*. The calculation of the size in cents of an interval proceeds in exactly the same way as our calculation earlier of the angle of the fifth in the circle. The only difference is that the result is not related to a circle but that the octave is now expressed in 1200 parts, since we proceed from the original division of the octave into twelve parts and then subdivide each twelfth into hundredths - hence the name *cents*. A cent is therefore a twelve-hundredth of an octave divided into twelve identically sized 'semitones', expressed as a logarithmic measure in relation to the number Two as the octave generating number. As a formula:

$$\text{Cent} = \frac{\log(\text{Proportion})}{\log(2)} \times 1200$$, or inversely:

$$\text{Proportion} = 2^{\text{Cent}/1200}$$

If we know this interrelation, we can also calculate the magnitude of the difference between the twelve fifths and the natural octave using this unit of measurement. We transpose by a further octave the value of 312/218 found earlier, so that we arrive in the same octave, and obtain the following:

$$\frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.01364326\ldots$$

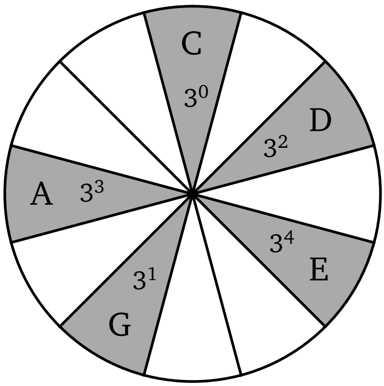
This value is the proportion - if, using the formula just stated, we translate it into cents, we get the following:

$$\frac{\log(1.01364326)}{\log(2)} \times 1200 = 23.46 \text{ Cent}$$

The difference between the 12 fifths and the pure octave is therefore around 23 cents - just under a quarter of a tempered semitone.

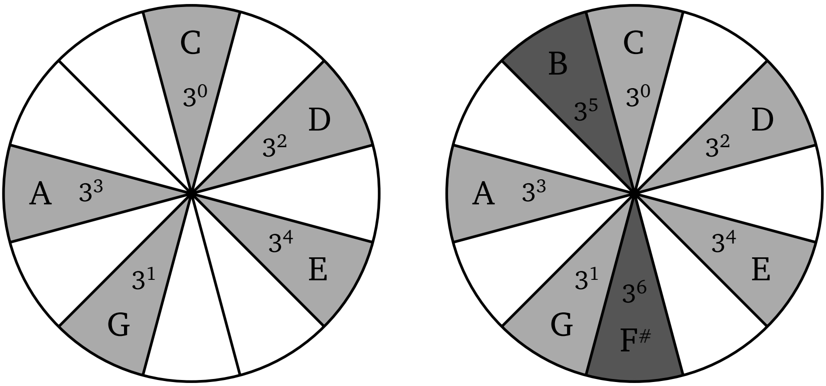
This value of 23 cents is not, however, something we perceive as a new tonal quality but as a pre-existing note being out of tune. This type of interval difference, which comes about through different ways of calculating a tone-step but without producing an interval quality of its own, is known as a *comma* - in this case, the *Pythagorean Comma*, this being the difference between 12 fifths and the fundamental. It is called Pythagorean because the tonal system based on the fifth 3:2 has always been known to us as the Pythagorean system.

But let us return to the old Chinese tone system - which in its origin and calculation is the same as the Pythagorean. We had established previously that the twelve tones came about through the continuous Three or fifth operation but we had not looked more closely at *which* tones arise successively. I will draw this again - for the purpose, I rotate the circle by half a segment, so that, instead of the line, the segment itself, i.e. the tone location of the fundamental, is at the top. If I now fill in the first five tones that arise from the steps of a fifth, I obtain this image:



We see that these five tones divide the tonal space relatively equally but asymmetrically. In fact, for their classical music, the Chinese halted their division of the tonal space at this point and used these five tones in their music. Let me play a short improvised melody using only these five tones:

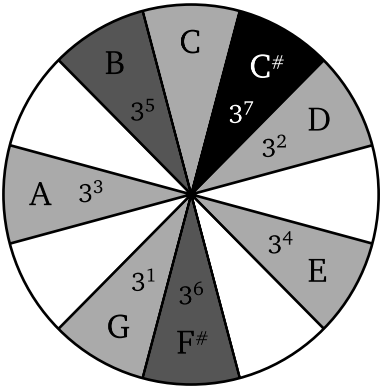
Although that is certainly not a Chinese composition, the tonal material gives us the impression that it somehow sounds Chinese. But how? It must be because the tonal space is divided up here in same way as in the graphic we saw just now. To make this clear, let us extend the succession of fifths - by adding a further two fifths only. If we extend the tonal space, then, we get this picture:

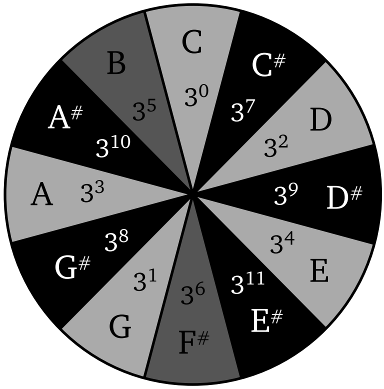


Whilst from the first four fifths we obtained five tones separated from one another by at least one or two tonal locations, the next step of a fifth as well as all subsequent ones yield tones directly adjacent to pre-existing ones. With five tones, we have therefore reached a frontier beyond which a completely new quality is encountered. How this quality expresses itself is something we will hear immediately if I play exactly the same short melody as before but change one note, introducing one of the tones yielded by the two most recent fifths.

I have only changed the penultimate tone, and we hear immediately how a completely new quality has entered - or how the old quality has collapsed - whichever way you like to look at it. How can we describe these qualities, this hearing experience? It is naturally very difficult in principle, if not even impossible, to capture such a qualitative experience in words. If I nonetheless attempt to do so, I can perhaps say that the five-step tone ladder or pentatonic evokes a sense of great openness- the melody stands to a certain extent freely in the space and has no desire to arrive anywhere. This experience is in accord with the old Taoist ideal of Chinese philosophy, the ideal of sanguine peace of mind. When the tone-steps are added that make the tonal space more cramped, the music acquires a stronger orientation - there's a pull towards certain tones. One could also say that the music has lost something of its free objectivity or that it's richer in terms of human subjectivity, that it delves more deeply into life. Through the agency of the newly added tones, tonal space has become more densely connected, the relationship between the tones is more strongly determined. We describe the quality arising from the seven fifth-tones as being characterized by 'leading tones', which bring to music a clearer orientation.

Let us see what happens if, after the first six steps of a fifth or seven notes, we continue by adding further fifths. We then get this picture:





When we go beyond there being seven notes, a situation arises for the first time after the next step of a fifth in which we have four directly adjacent tone locations, as we can see in the left-hand image, and in the right-hand one the tonal space is filled in completely. When there were only five or seven steps, we had in each case an asymmetric structuring of the tonal space. With twelve tones, however, it is no longer structured at all: it is fully symmetrical and therefore amorphous. What does that mean? A tonal system that is used musically is an organism - and an organism is distinguished by the fact that it has organs. One organ is different from another because it has a defined function; the heart, for example, has a different function to that of the lungs. In an asymmetrical tone system, all intervals relate in different ways to one another and therefore have different functions within the tone system - such an asymmetric tone system therefore has the characteristics of an organism. Where there are twelve tones, however, the functions of the various tone steps are no longer distinguishable - and for this reason the system no longer has the characteristics of an organism but is to be understood as a mere potential matrix of tonal locations for concrete tones moving within it.

This characteristic can be compared to a certain extent with the relationship between the zodiac and the planets: the zodiac gives only the twelve qualitative spatial directions in which the planets can reside - the tangible and describable realization of the time quality in each case, however, is only discernible from the concrete presence of the planets within this reference frame. The Chinese understood it exactly the same way with their tone system. I related at the very beginning that the principal court musician who was sent out to find the tone system returned with the twelve tones of the male and female phoenix and that these tones were cast as bells. Now, if the Chinese in their classical music never used more than five tones, what was the point of the remaining tones of the twelve-tone system?

In fact, the Chinese understood the division of the tonal space into twelve parts as an abstract tonal provision, from which was then assigned through the division into twelfths of the zodiac and the course of the year by the moon. From this supply, each month, five tones were taken and music was made with them. The next month, these tones were in a manner of speaking put back into the system and the next fundamental tone of the month selected upon which the five tones of that month would be built. But here a problem arose: the musical tones with which the concrete music was made were naturally the tones yielded by successive pure fifths 3:2. When these tones are realized in the system of twelve, we get a system that does not close, as we have already seen - the succession of fifths is not fully comprehended by the octave circle. If the concrete five tones used in music were identical with those from the system of twelve of the course of the annual round, the tonal circle as a whole would drift somewhat higher each year. So what did the Chinese do? They tempered the tone system. That means, they made each of the twelve fifths of the annual round smaller by a tiny bit, i.e. a twelfth of the comma that we calculated earlier, so that after doing the tour of the twelve fifths they returned to the starting point. It was not, however, with this tempered fifth system that the real music was made - it simply served as an abstract basis - an ideal matrix, so to speak - that was brought into concord with the annual round and from which the physical tones as pure intervals were then abstracted. We have, therefore, here in the old Chinese culture, hundreds of years before the thought ever occurred to us, for the first time a conscious approach to the problems of the tone system and the idea of temperament. What is important to note, here, however is that this temperament had no effect on the practical making of music but was simply the consequence of the recognition of the mathematical-ideal interrelation.

Let us now leave the old Chinese culture and look at how the same interrelations have presented themselves here in the Occident. On this subject, to begin with, there is not much new to be said. The same system that I have been describing here the whole time was also known to the Greeks. It is associated with the name of Pythagoras, who lived around 500 BC. It is said that it was he that described this tone system, and for around 2000 years, i.e. until around 1500 AD, it was the prevailing system in musical theory. Admittedly a number of Greek music theoreticians shortly after Pythagoras's time described further possibilities of tone systems, but these never became established in the practice of music over here. When I say 'in the practice of music over here', I mean as far back as we can trace it from sources relating to the theory and practice of music in Occidental culture, which is only to the Middle Ages. The music theoreticians of the Middle Ages cite almost exclusively the Greek philosophers and music theoreticians. About the *practice* of music in Ancient Greece, we know, alas, very little.

Our music prior to 1500 rested, then, like Chinese music, on the system of pure fifths, a succession of which yields a tone system. It differed, however, from the Chinese system in that instead of stopping when the fifth tone was reached, it comprised - for as far back as we can trace it - seven such tones. We have seen that these numbers, five and seven, are no arbitrary numbers; in dividing the tonal space specifically by these numbers a qualitative leap occurs, in consequence of the differing densities of related tones. We have also heard how these qualitative leaps present themselves psychologically and represent a difference in culture and mentality, with the Chinese favouring a pentatonic system, which, as we have established, conveys a sense of cheerfulness and tranquillity, and the Occidental system favouring a seven-step system, which imparts the experience of something more strongly coloured and shaped and that carries with it more of the element of intention and the will.

It was the number *Three*, then, that built the entire musical system - and the number *Three* is also the only one capable of doing so in this way. The other numbers merely add further differentiations to this system, further aspects, as we shall see shortly. Naturally it also played an important role in philosophy and theology that all of music rests upon a single principle of origin - and that furthermore it was the number *Three* was also a significant element from the point of view of the Christian Trinity. The idea that music arose from a single principle of creation naturally also meant that this type of music received encouragement and support from 'official' quarters, i.e. the theologians and philosophers.

Let us listen for a moment to an example of this music that was built upon the number *Three*. It comes from Pérotin of the Notre-Dame school in Paris in the first half of the 13th century.

Musical example: *"Viderunt omnes" by Pérotin sung by the Hilliard Ensemble, EMI Records*

This music seems to me a very clear realization of that which lies within this tone system that is built upon the number *Three*. With it, we leave the epoch of the *Three* - and let us listen perhaps to a further piece that represents a new quality that first appeared around 1500 - while the music of the preceding epoch is still fresh in our ears, so as better to compare their respective qualities.

Musical example: *"Sicut ovis ad occisionem" by Carlo Gesualdo, from "Tenebrae Responsories for Holy Saturday", sung by the Tallis Scholars, Gimell Records*

I believe it could be heard very clearly that this music has a quite different quality than that of the previous example - that here a totally new element has been added. Can one somehow express in words what is different here? I do not mean in a formal manner, the way a musicologist, for example, would do it if he were to compare both compositions. I mean the feeling that fundamentally distinguishes the one piece from the other. To get closer to it, one can perhaps only use images: For me, the earlier composition has something of a Gothic cathedral, that above all strives upwards, also something crystalline, as though the music were moving within a crystal lattice. Overall also somewhat more objectivity than the later composition - this later one has rather more of the organic, less of the mineral, about it. Less rigour. These are all naturally very odd descriptions of these musical impressions - moreover they are also very subjectively coloured, although I am only attempting to describe something that exists and grasp its character. There *is* a completely valid statement one could make that captures the difference between the two compositions - but it is so abstract, that I doubt whether it means anything to anyone. That statement would go like this: The difference lies in the number *Five*. In the first case, it is the absence, and in the second, the presence of the *Five* that makes us experience the music so differently in each case.

But after what I have already related, this statement perhaps does not seem quite as absurd to us any more, for we have already heard and experienced that the numbers do in fact "*do*" something. The number

Two created the phenomenon and the experience of the octave, and the number *Three* the experience of the fifth and, through it, gave a structure to tonal space. So it can be assumed that the other numbers, too, produce musically new qualities of experience. Let us perhaps ask ourselves first of all which of the other numbers are to be considered. So far, we've got to *Three*, so the next number would be *Four*. What does the *Four* do? We can answer that already without looking any further into the matter: it does nothing new - *Four* is twice *Two*, so it simply generates a further octave of the fundamental. We can also say much the same in the case of the *Six*: since *Six* is twice *Three*, six also produces nothing new but merely a repetition of the fifth, one octave higher. It is the same, though perhaps not quite as obvious, with the number *Nine*: although it may not represent another octave of an existing tone, it brings with it no new interval quality, since it is composed of 3 x 3 - and is therefore the fifth above the fifth - and we have therefore already encountered it as 3^2 in the series of fifths. We have established, then, that only those numbers that are not multiples of other numbers - i.e. only the prime numbers - are capable of contributing a new quality, a new musical experience. And the next prime number in the series is *Five*, so it was only logical that it should be the number *Five* that played the next greatest role in the history of music.

But this raises a further question: we have seen that through the repeated operation of the number *Three* the musical experience-space has already been completely filled with the twelve tone locations. What could anything new possibly add? But let us perhaps hear first what kind of interval experience it is that the number *Five* contributes. I set this number on the monochord - by which I mean, I take a fifth of the entire string, and for better comparison with the fundamental, I increase its length fourfold, i.e. octave the *Five* in the region of the fundamental. With a monochord string length of 120 cm, four-fifths makes 96 cm. I place the bridge at exactly this point and we listen: to the entire string together with the new tone produced by the *Five*. In actual fact this interval has a completely new quality that we experience still more clearly if we compare it with the fifth produced earlier through the ratio 2:3. Musically, we call this new experience the major third, or, if we think of the fundamental as being *C*, then this would be an *E*.

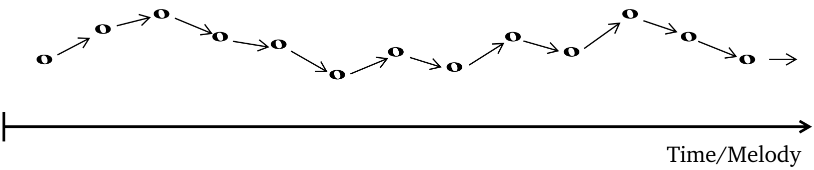
One might object at this point: What is new about that? We've already had a major third (or the *E* above a *C*) in the succession of fifths - all the tone locations have already been created by the *Three*! That is, of course, true - and yet not true. What has been added here is a new musical dimension: The third was already there before, but only as one tone of the scale among all the others, and now it is taking over a specific function that is totally new. Let us look at how that is meant. Previously the third came about as a consequence of the succession of 4 fifths, in a sense, as a side-effect and member of a series:

C	—	**G**	—	**D**	—	**A**	—	**E**
3^0		3^1		3^2		3^3		3^4
1		3		9		27		81

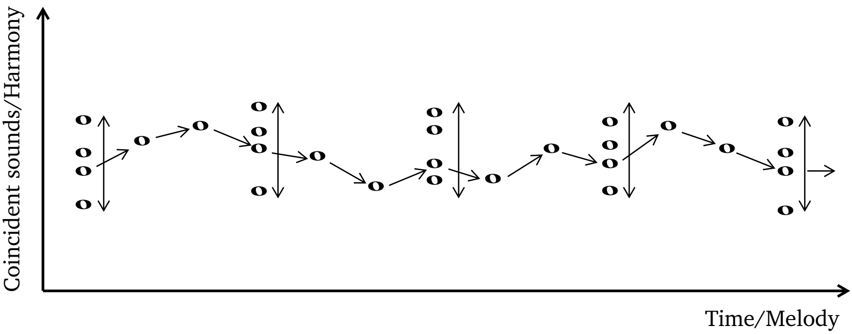
This series could continue like this - the third merely appears in it, occupying a place of no special importance with a profile no higher than that of the intervals around it. Not so the new third, the one we found through the number *Five*: it enjoys a special, self-defined relationship with the other intervals or tone numbers in its environment and has a very definite function in the organism of tone numbers. We can see this more clearly if we look more closely at the succession of natural numbers or the overtone series. For this purpose, I arrange the numbers as follows:

2		**:**		**3**
4	**:**	**5**	**:**	**6**
C		E		G

The numbers here that are above and below one another are in an octave relationship, so they are actually the same note. One could write all the natural numbers in this way, as they are ordered musically through the powers of two or octaves. 2 : 3 is a fifth, 4 : 6 is 'the same' fifth. The difference lies in the fact that the *Three* follows directly the *Two*, in the next octave level however the *Four* is followed by the *Five* - one could therefore say that the five "*introduces itself into the previously empty fifth*". Through it, it even defines its own function: to "fill" the fifth with a new experience quality. This function is most marked when tones coincide (i.e. sound simultaneously) - and if we listen to the way this 4:5:6 sounds, we realize that it is what we describe musically as a triad, a major triad. And this new function of the third, which arises from the number *Five*, first found acceptance among musicians around 1500 - generating a kind of 'euphoria of the third' that it is difficult for us today to imagine, since for us the triad is so familiar that we now often dismiss it from a musical standpoint as too trivial. The *Five* in fact brought to music a new dimension, and that term can be understood quite literally:



In earlier music, it was melody that played the key role - the course of the music in time, the succession of tones one after the other, the horizontal aspect. The new dimension contributed by the *Five* is the attention paid to the coincidence of sounds - the harmony, the vertical aspect:

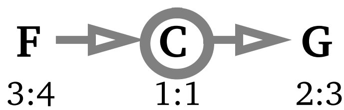


Of course, there was multipart music prior to this, but with the old multipart music each part, each line, stood for itself, and where notes coincided, what was regarded as important was to avoid dissonance as far as possible - a negative rule, so to speak. The harmonic intervals were a by-product of the melody-leading. Since the 'discovery' of the number *Five* - that is, the function of its essence - the vertical coincidence of tones, the harmony, was composed more consciously.

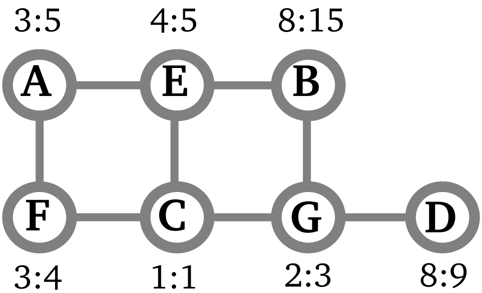
It is interesting to follow how this new quality 'crept into' music in the course of the 14th and 15th centuries. The harmony of the third came initially via folk music where it was already well established - but when the 'official' composers also began to arrange their music in this way, a bitter controversy broke out between the music theoreticians and the philosophers as to whether the *Five* should be allowed any place in music. That is understandable, because before that, of course, music could be explained as having a single principle of origin - that of the *Three* - and now the *Five* was throwing that very beautiful system into confusion. But by the year 1500 or so, the controversy had pretty much died down of its own accord - the *Five* was simply there.

But it brought not only philosophical but also totally practical problems with it and these could not be resolved so easily - to be more precise, these could not be resolved at all, and these problems are still there for us, and I would go so far as to say that these insoluble problems symbolize something quite fundamental in the essence of music. But on that subject one would need to philosophize at length - instead, I will go into the practical side, which brings back to our theme of tuning.

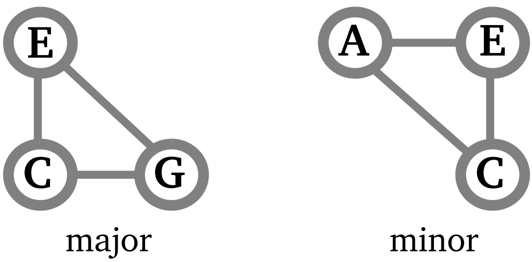
So we now have in our tone system, in a sense, two different types or families of tones, on the one side those related through the *Three*, and on the other, the relatives of the *Five*. And in the musical organism of the tone system, it can be determined with exactitude which notes are related to which. Let us look more closely at this now. First we have the fundamental, which for the sake of simplicity we have chosen to call *C*. Through the first and simplest relationship of the *Three*, the fundamental produces the fifth, *G*. In music as we experience it, however, the fundamental presents itself in two different ways: once actively, one might say, and once passively. One can experience this very nicely by playing what we call a simple cadence: the chord sequence *C* major, *G* major, *F* major and *C* major again, or tonic, dominant, subdominant, tonic. We hear how the chord based on the fundamental *C* in a sense 'produces' or 'brings forth' the dominant, and how after the next chord change, the *F* in turn does the same thing to the fundamental *C*. We perceive the fundamental or identification tone in other words the first time as producing and then a second time as being produced, as seeing and being seen, as active and passive, or in a yin and yang aspect. These three tones, which are related through fifths, are therefore not simply any arbitrary sequence but one that presents the fundamental in both its polar aspects:



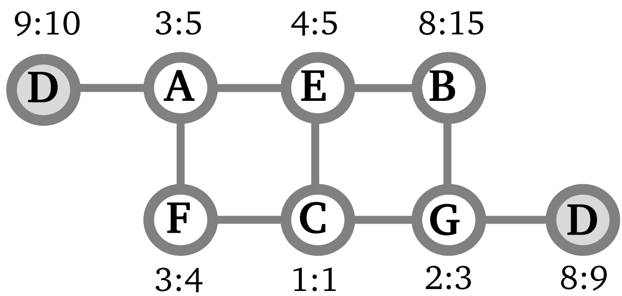
If we now introduce the new relationship of the *Five*, we must erect above each of these three elementary tones, the relationship 4:5:6 - i.e. a third 4:5 and a fifth 2:3 or 4:6. So as to be able to display this, too, graphically, we must introduce a second dimension: we have used the horizontal plane for the three or fifth relationships - so to show the *Five* or third relationships we must use the vertical plane:



So we have now above each of the starting tones, the relationship 4:5:6 - to complete the structure, I have added in the horizontal plane *D*, which is a fifth above *G*. The numbers arise from the defined ratios - this is easily confirmed by multiplying the ratios by one other - so, for example, the *B* is a third above the *G* (2:3) x (4:5) = 8:15. If we look to see which tones we have derived from this relationship structure, we will realize at once that in *C-D-E-F-G-A-B* we have the tones of our normal scale or the white keys on the piano. We could say, then, that this two-dimensional diagram represents the internal relationship structure of the organism of a scale, through which it in fact comes into being, and that this diatonic tone system as 'scale' only appears when the notes are arranged in a line. We also see in this diagram the characteristic structure of all major and minor chords:



So through this internal structure, each tone has so to speak found its own 'correct' or 'pure' location - and one might say that this is now a justly tuned tonality. And that would be true if each tone were only to appear in relation to the fundamental - but in our music it is the case that in harmonic structures, all tones appear in relationship with one another, by forming the triads of a key . Let us look at this more closely: Out of this tone provision, we can form major triads with *C, G* and *F* as their roots and minor triads with *A, E* and *D* as their roots (as well as a diminished triad with *B* as its root). These invariably have the structure illustrated in the diagram. If we look more closely at the *D* minor chord, however, we will realize that although we have its constituent tones *D-F-A*, the interval between the *D* and the *A* is not a natural fifth 2:3! If we listen to the chord formed by these notes, we will notice that it is so out of tune that it is completely unusable. We need, then, in this tone system another *D* - one, this time, that is a pure fifth from *A*. Represented graphically, it would look like this:



We have established therefore that we already need two different versions of one note even if we only wish to play in tune in *a single key.* So it is not even possible for *one* key to tune seven strings in such a way that they represent the pure tones of a key. In actual fact, in earlier times keyboard instruments were constructed upon which the key for *D* was split to make it possible to play both versions of this tone. That was obviously rather awkward for the player - and for this reason, the practice began of retuning the tones of the tone system in such a way that the *D* (as tuned) occupied a position halfway between the two 'pure' versions of the tone - hence the name *meantone* temperament applied to this system of tuning. To look more closely at this, we have to return to the point where we were comparing the two thirds, the one derived from the sequence of fifths and the one derived from the number *Five*.

So far, I have only shown that the two thirds are qualitatively different from one another, namely in having their origins in different numerical interrelations and also different functions within the musical organism. They also differ quantitatively, namely in the size of the interval produced by these two numerical interrelations. Let us look at this once again: the third derived from the succession of fifths, the Pythagorean third, is formed through 4 fifths; in numbers: 1, 3, 9, 27, 81. When brought into relation with the fundamental, i.e. a power of two, the 3 raised to the power four gives us the interval 64:81. The pure third has the ratio 4:5. How can we obtain the clearest comparison between the two intervals? Through the multiple extension of the interval 4:5 through the octaves: we obtain by this means 8:10, 16:20, 32:40 and finally 64:80. So we have 64:81 for the Pythagorean third and 64:80 for the pure third, or as the difference between the two, the ratio 80:81. This ratio is also a comma - it is called the *syntonic comma*. A short while ago, I set the pure third as 4/5 of 120 cm on the monochord - if we now wish to set the Pythagorean third also, we must take 64/81 of 120 cm, which gives us 94.81 cm. The string for the Pythagorean third is therefore some 12 mm shorter. If we now listen to this, we notice, that the Pythagorean is a great deal harsher sounding whilst the pure third is softer and fuses better. So it is no wonder that in earlier times the third was regarded as a dissonance and later as a consonance - formerly it was the Pythagorean, later the pure, third.

What would a temperament look like that ensured pure thirds but without forcing us to accept the phenomenon seen earlier of there being two different values for a single tone? In such a system, 4 fifths would again, as in the Pythagorean, have to produce an interval of a third, but we would have to detune each of these fifths by a small amount in order to ensure that four such jumps of a fifth were exactly equivalent to a pure third 4:5. In other words, we have to divide up the syntonic comma 80/81 equally between the 4 fifths. To obtain the requisite calculation, we have to find a number which, when multiplied by itself four times, comes to five - the result would be what we call the 'meantone fifth' (MF):

$$MF^4 = 5 \qquad \text{or} \qquad MF = \sqrt[4]{5} = 1.49534878\ldots$$

On the monochord, then, we obtain the meantone fifth by dividing 120 cm by 1.4953478... - that gives us 80.248 cm - in other words, some 2.5 mm more than a pure fifth at 80 cm. If we listen to that, we will notice that it is not too noisome and sounds, in fact, perfectly acceptable in the context of a major triad. In fact, this meantone temperament was the usual system of tuning throughout the entire Renaissance and even beyond. But before that it would have been unthinkable to detune the fifth - the holy fifth! - for the benefit of the third. That only shows how important and exhilarating the experience of the third was during this period. People wanted to be able to hear and enjoy this new experience in all its beauty.

There is only one snag with this tuning: if you go too far from the starting key, you run into a fifth that is absolutely unusable - it is known as the "wolf's fifth". This is easily understood if you remember that for this pure third tuning, the circle of fifths has to correspond to four pure thirds on top of one another - i.e. 5 x 5 x 5 =125. The next octave, or power of two, would be 128 here - we therefore have a comma of 125/128 as the difference - this is called *diësis*. It amounts to some 40 cents or 40% of a semitone and is musically no longer usable - we no longer perceive it as out of tune but quite simply as 'wrong'. In this meantone temperament, then, the tones *G#* and *Ab* do not coincide but are separated by this *diësis* and therefore completely different notes. At least, however, the "wolf's fifth" that results only appears in distant keys - so we no longer have the problem we found earlier when attempting within *one* single key to tune purely *all* the fifths and the thirds.

To be more exact, in the context of meantone temperament one cannot talk of a *circle* of fifths at all - the keys depart in two different directions and the distance between them grows ever greater. For this reason, instruments were built with far more than twelve keys per octave, with separate keys assigned, for example, to *G#* and *Ab*. But these instruments, obviously, were scarcely playable and never became established.

The more complex music became over time, the greater the desire to modulate into ever more remote keys and the less acceptable a system in which you were forever in danger of running into the 'wolf'. So people began to vary the temperament. The two poles of this variation were the Pythagorean system on the one hand with its pure fifths and meantone temperament on the other with its pure thirds. Only these two systems exemplified a principle in its pure form. The other tunings that emerged later were multiple variations with a single goal, which was, on the one hand, to maintain as far as possible the purity of the thirds, and on the other, to close the circle of fifths, making it possible to modulate freely between the keys. The general practice was to tune the keys around *C* in something like meantone temperament, but thereafter to make the fifths larger and consequently the thirds sharper. This is how the tunings of Werckmeister, Kirnberger and others originated - I will not go into the differences in detail in this context. A wanted side-effect of these tunings was to give different keys a different character - the central keys had a more harmonious and softer character, the more distant ones a sharper and harder one.

It was not until the mid 19th century that the system of tuning generally used today to tune pianos became established. This tuning is identical to the theoretical system of the Chinese of which the object was to close a full circle of fifths. This system of tuning is very simple - all it involves is reducing each fifth by a twelfth of the Pythagorean comma, thereby dividing the circle of fifths into intervals of identical size. As a result, you obtain fifths that come closer to pure ones than those of the meantone temperament - but also thirds that are closer to the Pythagorean third than to a pure one. So basically with our tuning of today we have lost precisely that which the meantone temperament of the Renaissance strove to achieve: the experience of the pure third. And that is why the equal temperament of today took so long to establish itself: it had been known about for a long time, but was not accepted because of the poor quality of its thirds and because of its uniformity.

Until quite recently, it was assumed that Bach's "Well-Tempered Clavier" was written for our own system of tuning - however, recent research has shown that for him our equally tempered tuning was not '*well*' tempered at all - because in it no key characteristics can be realized and the thirds are so bad. One should not, therefore, describe today's tuning as 'well-tempered' but rather as 'equally tempered' or uniform in terms of step size.

One interesting question is "why are we able today to accept the poor thirds of our piano tuning (and naturally all other instruments as well, since they are mainly oriented towards the piano) so much more easily than people during the Renaissance?" One answer that immediately suggests itself would be that our hearing has greatly deteriorated and is less sensitive. That is part of the answer, surely, but my own view is that this is not the most important reason. I believe the main reason is that the experience of the third in its actual function was a *novelty* around the year 1500 and that people wanted and felt compelled to experience it in its purest form. Since that time, the function of the third has become ever more deeply 'incarnated' in our sensibility and music - most powerfully in the classical era. Because this experience has become something we take for granted, all it needs is a reminder of the third, as afforded by our system of tuning, to activate the deep-rooted 'third' experience, which was not the case some 500 years ago. That does not mean, of course, that it would not be nicer to have pure thirds in our music.

If we have established that 500 years ago the experience of a new musical dimension was activated through the prime number *Five*, we can naturally ask ourselves whether the subsequent prime numbers do anything similar. The next prime number would be the *Seven* - which in actual fact is slowly making its presence felt in our time as a musically relevant element. Let us set the numbers 4:5:6:7 on the monochord - as the string lengths 4/4, 4/5, 4/6 and 4/7 of 120 cm - and listen to the result. We have here, to begin with, our familiar triad 4:5:6, and then comes the *Seven*. We have the impression of a seventh chord but this *Seven* sounds somewhat flatter than the one we are used to hearing in our music. If we think of a "seventh chord", then this is something that is already present in our previously existing music, where it is compounded of the proportion-numbers *Three* and *Five*. So it is no wonder that this new interval from the *Seven* sounds unfamiliar, and the new proportions cannot be represented by the old numerical ratios either - they have no place in the two-dimensional diagram of the tonal order we saw earlier, and to display the *Seven* ratios in this scheme, we would need to go into the third dimension - for the corresponding tonal order with the *Seven*, we therefore have a spatial model, a three-dimensional grid. We had established in the case of the five that it not only opened a new dimension in the display but also added a new dimension even in terms of content to music - in the case of the *Five*, it was the vertical dimension of the harmony as an independent element. It is to be assumed, then, that the *Seven*, too, will add not only new intervals but also in terms of content a new dimension to music. In what might this new dimension consist?

Naturally on the subject of the new function of the *Seven* in music, only cautious speculation is possible. In the case of the *Five*, we have 500 years of musical history to look back on, and it is relatively easy to make the corresponding assertions. How does the *Seven* express itself? First of all, we have already established that the chord that it extends resembles the dominant seventh chord with which we were already familiar. That does not mean, however, that it finds its function in the dominant seventh chord - just as the *Five* had no place in the linear music of the earlier epoch of the *Three*. The third as an interval already existed but not the third as a function. The *Seven* does not enter as a substitute the dominant seventh chord, which had already arisen organically as a product of the existing tonal order. In our normal major scale, no minor seventh appears as a relation to the fundamental - it appears in the key of C major only in the G major chord, which, through it, becomes a dominant chord that gravitates towards the tonic. The *Seven*, however, enters the tonic chord as something new, it is by its nature not a *dominant* but a *tonic* seventh. In fact, this new function is appearing in the music of today as if by its own volition - and indeed not so much primarily in the music of the 'serious' composers but in the blues and forms of jazz based on it. This is happening in a manner analogous to that in which the third found its way into the music of the establishment through folk music. I can imagine, then, that that which we experience in the blues as a new quality will in the course of time find its place quite naturally in modern music. But what is the actual qualitative new dimension of the *Seven* in music, in the sense that, for the *Three*, it was horizontal melody, and for the *Five*, vertical harmony? Here I can only speculate cautiously - but it strikes me that in our time the element of rhythm is gaining a quite independent significance - and just as there had been the simultaneous sounding of notes earlier, but that it was only through the function of the five that harmony acquired a qualitative importance of its own, perhaps rhythm, which was already present in music naturally, will acquire a new dimension of its own through the function of the *Seven*.

But these are naturally questions that we cannot resolve here and now, nor do we need to, because the development of music will provide the answers. In the symposium on Just Temperament that we will be holding tomorrow, the emphasis will primarily be upon practical matters, such as how just temperaments can be realized, and here the computer-controlled musical instruments of today are a boon. But we have now also seen that there can never be the criterion '*pure*' as an absolute measure, but that one must also be clear about the musical context in which an interval stands, about how it is *intended*, and what conception of '*pure*' arises from it. That is, in my view, a greater problem than the practical realization of just temperaments, and one that will doubtless provide material for further discussion.

About this article

All the articles are revised versions of lectures delivered in the context of events organized by the "Arbeitskreis Harmonik" (Study Group Harmonics) in the Freies Musikzentrum, Munich.

Peter Neubäcker: **What does 'just intonation' mean?**

Lecture delivered on the 12th June 1993. Since the lecture featured numerous demonstrations on the monochord among other things, the lecture was rewritten for the book.

Peter Neubäcker

Born on the 26th June 1953 near Osnabrück. After leaving school, a period of intensive searching for inner ways culminating in a three-year sojourn in a spiritual community. After that, apprenticeship as a violin-maker by Lake Constance. '79 onwards, own instrument-making workshop in Munich. There also three years' training as a non-medical practitioner, parallel to which, training in astrology. Development of astronomic-astrological materials that led to the founding of own publishing house that also forms my living basis (from a material point of view) along with courses in guitar-making and computer music at the Freies Musikzentrum in Munich.

For almost twenty years the emphasis of my work as regards content is the search for correspondences between structure and content - mainly in the field of music but also with reference to the natural sciences and philosophy - mostly and wherever possible formulated mathematically. From which it follows that I am engaged most of the time (alongside, or rather over and above, the work mentioned earlier) with musical-harmonic research and algorithmic composition and lead in terms of organization and content the "Arbeitskreis Harmonik" in Munich.

Originally appeared in:

Harmonik & Glasperlenspiel. Beiträge `93.
Verlag Peter Neubäcker & freies musikzentrum, München 1994